International Journal of Mathematics and
Computer Applications Research (IJMCAR)
ISSN(P 2249-6955; ISSN(E): 2249-8060
Vol. 5, Issue 3, Jun 2015, 75-80
TJPRC Pvt. Ltd.

LINE DOUBLE DOMINATION IN GRAPHS

M. H. MUDDEBIHAL[1] & SUHAS P. GADE[2]

[1]Department of Mathematics Gulbarga University, Gulbarga, Karnataka, India

[2]Department of Mathematics, Sangameshwar College, Solapur, Maharashtra, India

ABSTRACT

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is called a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set. A subset D^d of $V[L(G)]$ is a double dominating set of $L(G)$ if for every vertex $v \in V[L(G)]$, $|N[v] \cap D^d| \geq 2$, that is v is in D^d and has at least one neighbour in D^d or v is in $V[L(G)] - D^d$ and has at least two neighbours in D^d. The line double domination number $\gamma_{ddl}(G)$ is the minimum cardinality among all line double dominating sets of $L(G)$. In this paper many bounds on $\gamma_{ddl}(G)$ were obtained in terms of vertices, edges and other different parameters of G, but not the elements of $L(G)$, further we develop its relationship with other different domination parameters.

KEYWORDS: Line Graph, Dominating Set, Double Dominating Set, Double Domination Number

Subject Classification Number: AMS-05C69, 05C70.

1. INTRODUCTION

All graphs under consideration are finite undirected and loop-less without multiple edges. Let $G = (V, E)$ be a graph with vertex set V and edge set E. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G respectively. In general we use $< X >$ to denote the sub-graph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighbourhood of a vertex v, respectively. The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)(\Delta(G))$. A vertex of degree one is called an end vertex. Also $\beta_0(G)(\beta_1(G))$ is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G. $\chi(G)(\chi'(G))$ is the minimum n for which G has an n- vertices (n-edges) colourings. A line graph $L(G)$ is the graph whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. We begin with some standard definitions from domination theory. Let $G = (V, E)$ be a graph. A set D of vertices in a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number of G, denote by $\gamma(G)$ is the minimum cardinality of a dominating set. A set D subset of $V[L(G)]$ is said to be a dominating set of $L(G)$, if very vertex not in D is adjacent to a vertex in D of $L(G)$. The domination number of $L(G)$ is denoted by $\gamma[L(G)]$ is the minimum cardinality of a dominating set. A set D^d subset of $V[L(G)]$ is called a double dominating set of a $L(G)$ if

every vertex in $V[L(G)]$ is dominated by at least two vertices in S. Or a subset D^d of $V[L(G)]$ is a double dominating set of $L(G)$ if for every vertex $v \in V[L(G)], |N[v] \cap D^d| \geq 2$, that is v is in D^d and has at least one neighbour in D^d or v is in $V[L(G)] - D^d$ has at least two neighbours in D^d and is denoted by $\gamma_{ddl}(G)$. In this paper, many bounds on $\gamma_{ddl}(G)$ were obtained in terms of vertices, edges of G but not the member of $L(G)$ Also we establish line double domination of a line graph and express the results with other different domination parameters of G.

We need the following Theorem to prove our further results.

Theorem $A[1]$: Let G be a graph with $diam(G) = 2$ then$\gamma_t(G) \leq \delta(G) + 1$.

Theorem $B[4]$: If $G$is a graph without isolated vertices and $p \geq 3$ then $\gamma_{ss}(G) = \alpha_0(G)$.

Theorem $C[4]$: A non split dominating set D of G is minimal if and only if for each vertex $v \in D$ there exist a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$.

Theorem $D[2]$: For any connected (p, q) graph G, $\chi(G) \leq \Delta(G) + 1$.

Theorem $E[3]$: For any connected (p, q) graph G, $\left\lceil \frac{diam(G)}{3} \right\rceil \leq \gamma(G)$.

Observation 1: For any connected (p, q) graph G, $p - \gamma_{ddl}(G) \geq 1$.

2. Upper Bound for $\gamma_{ddl}(G)$:

We shall establish the upper bound for $\gamma_{ddl}(G)$ in terms of the vertices of G.

Theorem 1: For any connected (p, q) graph G, $\gamma_{ddl}(G) \leq p - 1$. Equality holds for P_3, C_3, C_4, C_5.

Proof: Suppose D^d is a double dominating set of $L(G)$. Then by definition of double domination, $|V[L(G)]| \geq 2$.Further by observation, $p - \gamma_{ddl}(G) \geq 1$. Clearly it follows that $\gamma_{ddl}(G) \leq p - 1$. Suppose G is isomorphic to P_3, C_3, C_4, C_5. Then in this case $|D^d| = p - 1$.

In Theorem 2, the upper bound for $\gamma_{ddl}(G)$ shall be expressed in terms of $\gamma(G)$ and vertices of G.

Theorem 2: For any connected (p, q) graph G, $\gamma_{ddl}(G) + diam(G) \leq p + \gamma(G)$.

Proof: Let $I = \{e_1, e_2, e_3, \dots, e_n\}$ subset of $E(G)$ be the minimal set of edges which constitutes the longest path between any two distinct vertices u, v $\in V(G)$ such that $dist(u, v) = diam(G)$. Furthermore let $D = \{v_1, v_2, \dots, v_i\}$ be any minimal dominating set of G and let $E = \{e_1, e_2, \dots, e_n\}$ be the set of edges of G. Now by definition of $L(G)$, $E(G) = V[L(G)]$. Let $D^d = \{u_1, u_2, \dots, u_k\}$ be the double dominating set of $L(G)$ such that$|N[u] \cap D^d| \geq 2 \forall u \in V[L(G)] - D^d$. It follows that $|D^d| \cup dist(u, v) \leq p \cup |D|$ and hence $\gamma_{ddl}(G) + diam(G) \leq p + \gamma(G)$.

Theorem3: For any connected (p,q) graph $G, \gamma_{ddl}(G) \le q$.

Proof: Suppose $H = \{u_1, u_2, \dots, u_n\}$ be the subset of $V[L(G)]$ and deg(u_i), $\forall\, u_i \in H$ has at least two. Then D_1 is subset of H form a minimal dominating set of $L(G)$. Further if $I = \{u_1, u_2, \dots, u_m\}$ be the set all end vertices in $L(G)$ then $I \cup H_1 = D^d$ where $H_1 \subseteq H$ form a double dominating set of $L(G)$ such that $|N[u] \cap D^d| \ge 2\ \forall\, u \in V[L(G)] - D^d$. Since $V[L(G)] = E(G) = q$, it follows that $|D^d| \le q$. Hence $\gamma_{ddl}(G) \le q$.

Theorem4: For any connected (p,q) graph $G, \gamma_{ddl}(G) + \gamma[L(G)] \le p + 2$.

Proof : Let D be the minimal dominating set of G. Now in $L(G)$, if $F = \{u_1, u_2, \dots, u_k\}$ be the set of all end vertices in $L(G)$ Then $F \cup H = D^d$, where $H \subseteq V[L(G)] - F$ forms a double dominating set of $L(G)$, such that $|N[u] \cap D^d| \ge 2\ \forall\, u \in V[L(G)] - D^d$ Since each vertex in $L(G)$ corresponds to the edges of G and each edge in G is incident to two vertices of G, it follows that $|D^d| \cup |D| \le p + 2$. Hence $\gamma_{ddl}(G) + \gamma[L(G)] \le p + 2$.

Theorem5: For any connected (p,q) graph $G, \gamma_{ddl}(G) \le p$.

Proof: Let D be any minimal dominating set of G. Further let $E = \{e_1, e_2, \dots, e_n\}$ be the set of all edges which are incident to the vertices of G. Now by definition of line graph, $V[L(G)] = E(G)$. Suppose $I = \{u_1, u_2, \dots, u_i\}$ be the set of all end vertices in $L(G)$, then $I \cup H = D$ where H

Subset of F, forms a double dominating set of $L(G)$ such that $|N[u] \cap D^d| \ge 2\ \forall\, u \in V[L(G)] - D^d$. Clearly $|D^d| = |I \cup H| \le p$. It follows that $\gamma_{ddl}(G) \le p$.

Theorem 6: For any connected (p,q) graph $G, \gamma_{ddl}(G) + \chi(G) \le p + \Delta(G)$.

Proof: By Theorem1 and by Theorem D, clearly it follows that $\gamma_{ddl}(G) + \chi(G) \le p + \Delta(G)$.

Theorem 7: For any connected (p,q) graph $G, \gamma_{ddl}(G) + \gamma(G) \le p + \left\lceil \frac{\alpha_0}{2} \right\rceil$. Equality holds for C_4.

Proof : Let $B = \{v_1, v_2, \dots, v_k\}$ be the minimum set of vertices which covers all the edges of G such that $|B| = \alpha_0$. Further D be a γ-set of G. Let $E = \{e_1, e_2, \dots, e_n\}$ be the set of all edges of G.

Now by definition of line graph $L(G), E(G) = V[L(G)]$. Suppose $I = \{u_1, u_2, \dots, u_k\}$ be the set of all end vertices in $L(G)$, then $|I \cup H| = D^d$ where $H \subseteq E$, forms a double dominating set of $L(G)$ such that $|N[u] \cap D^d| \ge 2\ \forall\, u \in V[L(G)] - D^d$. It follows that $2\{|H \cup I| \cup |D|\} - |B| \le 2p$ and hence $\gamma_{ddl}(G) + \gamma(G) \le p + \left\lceil \frac{\alpha_0}{2} \right\rceil$. Suppose G is isomorphic to C_4. Then in this case, $|B| = 2$ and $\alpha_0(G) = 2 = \alpha(G)$ Clearly, $\gamma_{ddl}(G) + \gamma(G) \le p + \left\lceil \frac{\alpha_0}{2} \right\rceil$.

Theorem 8: For any connected (p,q) graph G $\gamma_{ddl}(G) \leq \gamma_s(G) + \gamma_{ss}(G)$.

Proof: By Theorem 12 and Theorem 13 the result follows.

3. Lower Bound for $\gamma_{ddl}(G)$:

Theorem 9: For any connected (p,q) graph G, $\left\lceil \frac{p}{\Delta(G)} \right\rceil \leq \gamma_{ddl}(G)$.

Proof: Let $D = \{v_1, v_2, \dots, v_k\}$ be any minimal dominating set of G and let $F = \{e_1, e_2, \dots, e_i\}$ be the set of edges which are incident with the vertices of G. Now by the definition of $L(G)$,

$F \subseteq V[L(G)]$. Clearly $D^d = \{u_1, u_2, \dots, u_k\} \subseteq F$ in $L(G)$ forms the double dominating set of $L(G)$ such that $|N[u] \cap D^d| \geq 2 \ \forall u \in V[L(G)] - D^d$. Further, suppose $C = \{v_1, v_2, \dots, v_n\} =$ be the set of all non end vertices in G, then there exists at least one vertex v of maximum degree

$\Delta(G)$ in C, such that $|D^d|.\Delta(G) \geq p$. It follows that $\gamma_{ddl}(G) \geq \left\lceil \frac{p}{\Delta(G)} \right\rceil$.

Theorem 10: If every non end vertices of a tree is adjacent to at least one end vertices, then $\gamma_{ddl}(G) \geq p - m$. Where m is the number of end vertices in T.

Proof: Let T be a tree. If $diam(T) \geq 3$ and $S = \{v_1, v_2, \dots, v_m\}$ be the set of all end vertices of T with $|S| = m$ and $d(v_i) = 1, 1 \leq i \leq m$. Let $E = \{e_1, e_2, \dots, e_i\}$ be the edge set of T. Now by the definition line graph $L(G)$, $E(G) = V[L(G)]$ Suppose $I = \{u_1, u_2, \dots, u_k\}$ be the set of all end vertices in $L(G)$, then $I \cup H = D^d$ where $H \subseteq E$, forms a double dominating set of $L(G)$ such that $|N[u] \cap D^d| \geq 2 \forall\ u\ V[LG)] - D^d$. Since for any tree T, $q = p - 1$, it follows that $|D^d| \geq p - |S|$ and hence, $\gamma_{ddl}(G) \geq p - m$.

Theorem 11: For any connected (p,q) graph G, $\gamma_t(G) \leq \gamma_{ddl}(G)$.

Proof: Let $v \in V(G)$ and $\deg(v) = \delta(G)$. Since $diam(G) = 2$, then by Theorem A the dominating set $D, |D| \leq \delta(G) + 1$. Therefore, $\gamma_t(G) \leq \delta(G) + 1$. Suppose for any connected graph with $diam(G) \geq 2$, again by Theorem A , $|D| \geq \delta(G) + 1$. Hence $\gamma_t(G) \geq \delta(G) + 1$. Now let D^d be a double dominating set of $L(G)$ such that $|N[u] \cap D^d| \geq 2 \ \forall u \in V[L(G)] - D^d$. Again by Theorem A, $|D^d| \geq \delta[L(G)] + 2$. Clearly it follows that $\gamma_{ddl}(G) \geq \delta[L(G)] + 2$. Hence $\gamma_t(G) \leq \gamma_{ddl}(G)$.

Theorem 12: For any connected (p,q) graph G, $\gamma_s(G) \leq \gamma_{ddl}(G)$.

Proof: Let S be a maximum independent set of vertices in G. Then there exists a set S_1 subset of S such that S_1 has at least two vertices and every vertex in S_1 is adjacent to some vertex in $V - S_1$. Hence $V - S_1$ is a split dominating set of G. Therefore $|V - S_1| \leq |S|$. Hence $\gamma_s(G) \leq \beta_0$. Now let D^d be a double dominating set in $L(G)$ such

that $|N[u] \cap D^d| \geq 2\ \forall\ u \in V[L(G)] - D^d$. Since $E(G) = V[L(G)]$. and let S^I be a maximum independent set of L(G). Then every vertex in S^I is adjacent to some vertex in $V[L(G)] - D^d$, such that $|N[v] \cap S^I| \geq 1\ \forall\ v \in V[L(G)]$. Clearly, $|N[v] \cap S^I| \leq |N[v] \cap D^d|$ it follows that $\beta_0[L(G)] \leq \gamma_{ddl}(G)$. Hence $\gamma_s(G) \leq \gamma_{ddl}(G)$.

Theorem 13: For any connected (p, q) graph $\gamma_c(G) \leq \gamma_{ddl}(G)$.

Proof: Let v be a vertex of maximum degree $\Delta(G)$. Then v is adjacent to $N(v)$ vertices such that $\Delta(G) = N(v)$. Hence $V - N(v)$ is a dominating set. Let D be a connected dominating set of G such that $D \leq V - \Delta(G)$. Therefore $|D| \leq |V - N(v)|$. Hence $\gamma_c(G) \leq p - \Delta(G)$. Now, let D^d be a double dominating set of $L(G)$ such that $|N[v] \cap D^d| \geq 2\ \forall\ u \in V[L(G)] - D^d$. Also $D^d \geq V - \Delta[L(G)]$. Therefore $|D^d| \geq |V - \Delta(G)|$, it follows that $\gamma_{ddl}(G) \geq p - \Delta[L(G)]$. Hence $\gamma_c(G) \leq \gamma_{ddl}(G)$.

Theorem 14: For any connected (p, q) graph G, $\gamma_{ss}(G) \leq \gamma_{ddl}(G)$.

Proof: Let S be a maximum independent set of vertices in G. Then $V - S$ is a strong split dominating set of G. Since S is maximum, $V - S$ is minimum. Thus $\gamma_{ss}(G) = \alpha_0(G)$. Now let D^d be a double dominating set in $L(G)$. Since $E(G) = V[L(G)]$. Let S^I be a maximum independent set of $L(G)$. Then $V[L(G)] - S^I$ is minimum and $|V[L(G)] - S^I| \leq |D^d|$. Clearly it follows that $\alpha_0[L(G)] \leq \gamma_{ddl}(G)$. Hence $\gamma_{ss}(G) \leq \gamma_{ddl}(G)$.

Theorem 15: For any connected (p, q) graph $\gamma_{ns}(G) \leq \gamma_{ddl}(G)$.

Proof : By Theorem $[C]$, a non-split dominating set D of G is minimal if and only if for each vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$. Therefore $|N(u) \cap D| = 1$. Now let D^d be a double dominating set of $L(G)$ such that $|N[u] \cap D^d| \geq 2\ \forall\ u \in V[L(G)] - D^d$. From the above, if for each vertex $v \in D^d$ then there exists a vertex $u \in V - D^d$ such that $N(u) \cap D^d = \{v_i, v_j\}$ for $i \neq j$ and $1 \leq i, j \leq n$. Therefore $|N(u) \cap D^d| = 2$. It is clear that $|N(u) \cap D| \leq |N(u) \cap D^d|$. Hence $\gamma_{ns}(G) \leq \gamma_{ddl}(G)$.

Theorem 16: For any connected (p, q) graph $G, \gamma(G) \leq \gamma_{ddl}(G)$.

Proof : Let $E = \{e_1, e_2, \ldots, e_n\}$ be the set of edges of G. Let $D = \{v_1, v_2, \ldots, v_k\}$ be any minimal dominating set of G such that for every vertex $v \in V(G) - D$ such that $|N[v] \cap D| \geq 1$. Now by definition of $L(G)$, $V[L(G)] = E(G)$, let $D^d = \{u_1, u_2, \ldots, u_i\}$, $1 \leq i \leq n$, in $L(G)$, forms the double dominating set of $L(G)$, such that $|N[u] \cap D^d| \geq 2\ \forall\ u \in V[L(G)] - D^d$. It follows that $|D| \leq |D^d|$ and

hence $\gamma(G) \leq \gamma_{ddl}(G)$.

Theorem 17: For any connected (p,q) graph G, $\left\lceil \frac{diam(G)}{3} \right\rceil \leq \gamma_{ddl}(G)$.

Proof: By Theorem $[E]$ and Theorem $[16]$ the result follows.

REFERENCES

1. Gangadharappa D. B. And A. R. Desai, Some bounds on Domination of a graph, J.Comp. and Math.Sci. Vol. 2(2), 234-242(2011).
2. F. Harary, “Graph Theory”, Adison Wesley, Reading Mass (1972)
3. T.W.Haynes, S.T. Hedetniemi and P.J. Slater, “Fundamentals of Domination in Graphs”. Marcel Dekker, Inc; New York, (1998).
4. V. R. Kulli, “Theory of Domination in Graphs”, Vishwa International Publications, India, (2010).